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Application of wave packet transform to time dependent Schrödinger equations

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Abstract

In this note, we determine the wave front sets of solutions to Schrödinger equations of a harmonic oscillator with sub-quadratic perturbation by using the representation of the Schrödinger evolution operator of a harmonic oscillator introduced in [11] via the wave packet transform. In the previous paper [14], the authors have studied the wave front sets for Schrödinger equations with sub-quadratic potential.

1 Introduction

In this note, we consider the following initial value problem of the Schrödinger equations of a harmonic oscillator with sub-quadratic perturbation,

$$\begin{cases} i\partial_t u + \frac{1}{2}\triangle u - \frac{1}{2}|x|^2 u - v(t,x)u = 0, & (t,x) \in \mathbb{R}\times\mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{1}$$

where $i=\sqrt{-1}$, $u:\mathbb{R}\times\mathbb{R}^n \to \mathbb{C}$, $\triangle = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ and $v(t,x)$ is a real valued function.

The aim of this note is to determine the wave front sets of solutions to the Schrödinger equations (1) of a harmonic oscillator with sub-quadratic perturbation $v(t,x)$ by using the representation of the Schrödinger evolution operator of a free particle introduced in [11] via the wave packet transform which is defined by A. Córdoba and C. Fefferman [1]. Wave packet transform is almost the same transform as the ones which are known as short time Fourier transform ([7]) and F. B. I. transform ([3]).

We assume the following assumption on the perturbation $v(t,x)$.

Assumption 1.1. $v(t,x)$ is a real valued fuction in $C^\infty(\mathbb{R}\times\mathbb{R}^n)$ and there exists a real number ρ with $0\le\rho<2$ such that for all multi-indices α, there exists $C_\alpha>0$ satisfying

$$|\partial_x^\alpha v(t,x)| \le C_\alpha(1+|x|)^{\rho-|\alpha|}$$

for all $(t,x) \in \mathbb{R} \times \mathbb{R}^n$.

Let $\varphi \in \mathcal{S}(\mathbb{R}^n)\backslash\{0\}$ and $f \in \mathcal{S}'(\mathbb{R}^n)$. We define the wave packet transform $W_\varphi f(x,\xi)$ of f with the wave packet generated by a function φ as follows:

$$W_\varphi f(x,\xi) = \int_{\mathbb{R}^n} \overline{\varphi(y-x)} f(y) e^{-iy\xi} dy, \quad x,\xi \in \mathbb{R}^n.$$

The authors have given a representation of the Schrödinger evolution operator of a free particle in the previous paper [11]:

$$W_{\varphi(t)}u(t,x,\xi) = e^{-\frac{i}{2}t|\xi|^2} W_{\varphi_0}u_0(x-\xi t,\xi), \tag{2}$$

where $\varphi(t) = \varphi(t,x) = e^{i(t/2)\triangle}\varphi_0(x)$ with $\varphi_0(x) \in \mathcal{S}(\mathbb{R}^n)\backslash\{0\}$ and $W_{\varphi(t)}u(t,x,\xi) = W_{\varphi(t,\cdot)}[u(t,\cdot)](x,\xi)$. This representation is introduced in the section 3. In the following, we often use this convention $W_{\varphi(t)}u(t,x,\xi) = W_{\varphi(t,\cdot)}[u(t,\cdot)](x,\xi)$ for simplicity.

In order to state our results precisely, we prepare several notations. Let $b = (2-\rho)/4$. For $\varphi_0(x) \in \mathcal{S}(\mathbb{R}^n)\backslash\{0\}$, we put $\varphi(t,x) = e^{i(t/2)(\triangle-|x|^2)}\varphi_0(x)$ and $\varphi_\lambda(t,x) = e^{i(t/2)(\triangle-|x|^2)}\varphi_{0,\lambda}(x)$ with $\varphi_{0,\lambda}(x) = \lambda^{nb}\varphi_0(\lambda^b x)$. For $(x_0,\xi_0) \in \mathbb{R}^n \times \mathbb{R}^n\backslash\{0\}$, we call a subset $V = K \times \Gamma$ of $\mathbb{R}^{2n}$ a conic neighborhood of (x_0,ξ_0) if K is a neighborhood of x_0 and Γ is a conic neighborhood of ξ_0 (i.e. $\xi \in \Gamma$ and $\alpha > 0$ implies $\alpha\xi \in \Gamma$). For $\lambda > 0$ and $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$, let $x(s;t,x,\lambda\xi)$ and $\xi(s;t,x,\lambda\xi)$ be the solutions to

$$\begin{cases} \dot{x}(s) &= -\xi(s), \quad x(t) = x, \\ \dot{\xi}(s) &= x(s) + \nabla v(s,x(s)), \quad \xi(t) = \lambda\xi. \end{cases} \tag{3}$$

The following theorem is our main result.

Theorem 1.2. *Let $u_0(x) \in L^2(\mathbb{R}^n)$ and $u(t,x)$ be a solution of (1) in $C(\mathbb{R};L^2(\mathbb{R}^n))$. Then under the assumption 1.1, $(x_0,\xi_0) \notin WF(u(t,\cdot))$ if and only if there exists a conic neighborhood $V = K \times \Gamma$ of (x_0,ξ_0) such that for all $N \in \mathbb{N}$, for all $a \geq 1$ and for all $\varphi_0(x) \in \mathcal{S}(\mathbb{R}^n)\backslash\{0\}$, there exists a constant $C_{N,a,\varphi_0} > 0$ satisfying*

$$|W_{\varphi_\lambda(-t)}u_0(x(0;t,x,\lambda\xi),\xi(0;t,x,\lambda\xi))| \leq C_{N,a,\varphi_0}\lambda^{-N}$$

for $\lambda \geq 1$, $a^{-1} \leq |\xi| \leq a$ and $(x,\xi) \in V$.

Remark 1.3. $W_{\varphi_\lambda(-t)}u_0(x,\xi)$ is the wave packet transform of $u_0(x)$ with a wave packet $\varphi_\lambda(-t,x)$.

Remark 1.4. The authors have determined the wave front sets of solutions to Schrödinger equations of a free particle and a harmonic oscillator in [12] and have determined the wave front sets of solutions to Schrödinger equations with subquadratic potential in [14].

Remark 1.5. In one space dimension, K. Yajima [25] shows that the fundamental solution of Schrödinger equations with super quadratic potential has singularities everywhere.

Corollary 1.6. *If $\rho < 1$, then $(x_0, \xi_0) \notin WF(u(t,\cdot))$ if and only if there exists a conic neighborhood $V = K \times \Gamma$ of (x_0, ξ_0) such that for all $N \in \mathbb{N}$, for all $a \geq 1$ and for all $\varphi_0(x) \in \mathcal{S}(\mathbb{R}^n)\backslash\{0\}$, there exists a constant $C_{N,a,\varphi_0} > 0$ satisfying*

$$|W_{\varphi_\lambda(-t)} u_0(x\cos t - \lambda\xi \sin t, \lambda\xi \cos t + x \sin t)| \leq C_{N,a,\varphi_0} \lambda^{-N}$$

for $\lambda \geq 1$, $a^{-1} \leq |\xi| \leq a$ and $(x, \xi) \in V$.

Microlocal characterization of the singularities of generalized functions was studied firstly by M. Sato, J. Bros and D. Iagolnitzer and L. Hörmander independently around 1970. The notion of wave front set is introduced by L. Hörmander in 1970 ([9]). He has shown that the wave front set of solutions to the linear hyperbolic equations of principal type propagates along the null bicharacteristics([10]).

The singularities of solutions to Schrödinger equations have been treated microlocally by R. Lascar [16], C. Parenti and F. Segala [22] and T. Sakurai [23].

Since the Schrödinger operator $i\partial_t + \frac{1}{2}\triangle$ commutes $x + it\nabla$, the solutions to Schrödinger equations become smooth for $t > 0$ if the initial data decay at infinity. In [2], W. Craig, T. Kappeler and W. Strauss have shown for solutions that for a point $x_0 \neq 0$ and a conic neighborhood Γ of x_0, $\langle x \rangle^r u_0(x) \in L^2(\Gamma)$ implies $\langle \xi \rangle^r \hat{u}(t, \xi) \in L^2(\Gamma')$ for a conic neighborhood Γ' of x_0 and for $t \neq 0$, though they have considered more general operators. Several mathematicians have studied in this direction ([4], [5], [18], [20], [21]).

A. Hassell and J. Wunsch [8] and S. Nakamura [19] determine the wave front set of the solution by the information of the initial data. Hassell and Wunsch have treated the singularities as "scattering wave front set" which is introduced by himself. In the case that the electric potential is sub-quadratic, Nakamura has shown that for a solution $u(t, x)$, $(x_0, \xi_0) \notin WF(u(t,\cdot))$ if and only if there exists a C_0^∞ function $a(x, \xi)$ in $\mathbb{R}^{2n}$ with $a(x_0, \xi_0) \neq 0$ such that $\|a(x + tD_x, hD_x)u_0\| = O(h^\infty)$ as $h \downarrow 0$.

For Schrödinger equations with harmonic oscillator or perturbed harmonic oscillators, S. Zelditch [27] determines the singular support of the fundamental solution $k(t, x, y)$ in the case that $v \equiv 0$, which shows that

$$\operatorname{sing\,supp} k(t, \cdot, y) = \begin{cases} \emptyset & \text{if } t \neq m\pi \\ (-1)^m y & \text{if } t = m\pi. \end{cases} \tag{4}$$

L. Kapitanski, I. Rodnianski and K. Yajima [15] have shown that (4) holds for $\rho < 1$ and may fail for $\rho = 1$. K. Yajima [26] has shown that if the Hessian of $a(x)$ is positive definite, then $\operatorname{sing\,supp} k(t, \cdot, y) = \emptyset$ for $t \neq 0$. S. Mao and S. Nakamura [17] have determined the wave front sets of the solutions of (1) in the case that $\rho < 1$. J. Wunsch [24] has studied regularity of the solution on scattering manifold in the case that $\rho \leq 1$. T. Ōkaji [21] has investigated the wave front set of the solutions for $t = m\pi$ with an integer m in the case that $v \equiv 0$.

2 Preliminaries

In this section, we introduce the definition of wave front set $WF(u)$ and the characterization of wave front set by G. B. Folland [6].

Definition 2.1 (Wave front set). For $f \in \mathcal{S}'(\mathbb{R}^n)$, we say $(x_0, \xi_0) \notin WF(f)$ if there exist a function $\chi(x)$ in $C_0^\infty(\mathbb{R}^n)$ with $\chi(x_0) \neq 0$ and a conic neighborhood Γ of ξ_0 such that for all $N \in \mathbb{N}$ there exists a positive constant C_N satisfying

$$|\widehat{\chi f}(\xi)| \leq C_N(1 + |\xi|)^{-N}$$

for all $\xi \in \Gamma$.

To prove Theorem 1.2, we use the following characterization of the wave front set by G. B. Folland [6]. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)\backslash\{0\}$. For fixed b with $0 < b < 1$, we put $\varphi_\lambda(x) = \lambda^{nb/2}\varphi(\lambda^b x)$.

Proposition 2.2 (G. B. Folland [6, Theorem 3.22], T. Ōkaji [20, Theorem2.2] and [13]). *For $f \in \mathcal{S}'(\mathbb{R}^n)$, we have $(x_0, \xi_0) \notin WF(f)$ if and only if there exist a conic neighborhood K of x_0 and a conic neighborhood Γ of ξ_0 such that for all $N \in \mathbb{N}$ and for all $a \geq 1$ there exists a constant $C_{N,a} > 0$ satisfying*

$$|W_{\varphi_\lambda} f(x, \lambda\xi)| \leq C_{N,a}\lambda^{-N}$$

for $\lambda \geq 1$, $x \in K$ and $\xi \in \Gamma$ with $a^{-1} \leq |\xi| \leq a$.

Remark 2.3. G. B. Folland [6] has shown that the conclusion follows if the wave packet φ is an even and nonzero function in $\mathcal{S}(\mathbb{R}^n)$ and $b = 1/2$. In T. Ōkaji [20], the proof of Proposition 2.2 for $b = 1/2$ is given if φ satisfies $\int x^\alpha \varphi(x)dx \neq 0$.

Remark 2.4. G. B. Folland [6] and T. Ōkaji [20] have proved for $b = 1/2$. It is easy to extend for $0 < b < 1$.

3 Simple examples

Our idea is to use a time dependent wave packet. When we consider a partial differential equation (A) of order 1 in time and of order 2 in space such as Schrödinger equation, we can transform the equation (A) to a partial differential equation (B) of order 1 with respect to all variables (t, x, ξ) in $\mathbb{R}^{2n+1}$ with remainder terms via the wave packet transform with the suitable time dependent wave packet. The equation (B) can be solved or be tranformed to an integral equation, by which we can study the solution of (A) (See the figure below).

$$\begin{tikzcd}
\text{P.D.E. of 2nd order(A)} \arrow[r, "W_{\varphi(t)}"] & \text{P.D.E of 1st order} + \text{remainder}(B) \arrow[d, "\text{Solve}"] \\
\text{Studying sol. of (A) by sol. of (B)} & \text{sol. of (B) or Integral Eq.} \arrow[l]
\end{tikzcd}$$

To illustrate the idea, we give two examples.

Example 3.1 (Free particle). Consider the following initial value problems(Schrödinger equation of a free particle):

$$\begin{cases} i\partial_t u + \frac{1}{2}\triangle u = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{5}$$

Let $\varphi(t,x) = e^{\frac{i}{2}t\triangle}\varphi_0$ with $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)\backslash\{0\}$. (5) is transformed via the wave packet transform with the wave packet $\varphi(t,x)$ to

$$\begin{cases} (i\partial_t + i\xi\cdot\nabla_x - \frac{1}{2}|\xi|^2)W_{\varphi(t)}u(t,x,\xi) = 0, \\ W_{\varphi(0)}u(0,x,\xi) = W_{\varphi_0}u_0(x,\xi). \end{cases} \tag{6}$$

Solving (6), we have

$$W_{\varphi(t)}u(t,x+\xi t,\xi) = W_{\varphi_0}u_0(x,\xi), \tag{7}$$

or

$$W_{\varphi(t)}u(t,x,\xi) = W_{\varphi_0}u_0(x-\xi t,\xi). \tag{8}$$

Remark 3.2. The representation of a solution (7) for a free particle is natural as phisical point of view, because $(x+\xi t,\xi)$ is the classical orbit of a free particle starting from (x,ξ) in the phase space $\mathbb{R}^{2n}$.

Example 3.3 (Harmonic Oscillator). Consider the following Schrödinger equation of a harmonic oscillator:

$$\begin{cases} i\partial_t\varphi + \frac{1}{2}\triangle\varphi - \frac{1}{2}|x|^2\varphi = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}^n, \\ \varphi(0,x) = \varphi_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{9}$$

Let $\varphi(t,x) = e^{\frac{i}{2}t(\triangle - |x|^2)}\varphi_0$ with $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)\backslash\{0\}$. (9) is transformed via the wave packet transform with the wave packet $\varphi(t,x)$ to

$$\begin{cases} (i\partial_t + i\xi\cdot\nabla_x - ix\cdot\nabla_\xi - \frac{1}{2}(|\xi|^2 - |x|^2))W_{\varphi(t)}u(t,x,\xi) = 0, \\ W_{\varphi(0)}u(0,x,\xi) = W_{\varphi_0}u_0(x,\xi). \end{cases} \tag{10}$$

Solving this first order partial differential equation (10), we have

$$W_{\varphi(t)}u(t,x,\xi) = e^{-\frac{i}{2}\int_0^t(|\xi(t-s)|^2 - |x(t-s)|^2)ds}W_{\varphi_0}u_0(x(t),\xi(t)),$$

where

$$\begin{cases} x(t) &= x\cos t - \xi\sin t, \\ \xi(t) &= \xi\cos t + x\sin t. \end{cases}$$

4 Proof of Theorem 1.2

In this section, we give a brief proof of Theorem 1.2.

Proof of Theorem 1.2. The initial value problem (1) is transformed via the wave packet transform with the wave packet generated by $\varphi(t,x)$ in Example 3.3 to

$$\begin{cases} \left(i\partial_t + i\xi\cdot\nabla_x - i(x+\nabla_x v(t,x))\cdot\nabla_\xi - \frac{1}{2}|\xi|^2 - \widetilde{V}(t,x)\right)\times \\ \qquad\qquad W_{\varphi(t)}u(t,x,\xi) = Ru(t,x,\xi), \\ W_{\varphi(0)}u(0,x,\xi) = W_{\varphi_0}u_0(x,\xi), \end{cases} \tag{11}$$

where $\widetilde{V}(t,x) = -\frac{1}{2}|x|^2 + v(t,x) - \nabla_x v(t,x)\cdot x$ and

$$Ru(t,x,\xi) = \sum_{|\alpha|=2}\int \overline{\varphi(t,y-x)}\partial_x^\alpha v(t,x,y)(y-x)^\alpha u(t,y)e^{-i\xi y}dy$$

with $\partial_x^\alpha v(t,x,y) = \frac{1}{\alpha!}\int_0^1 \partial_x^\alpha v(t,x+\theta(y-x))(1-\theta)d\theta$. Solving (11), we have the integral equation

$$\begin{aligned} W_{\varphi(t)}u(t,x,\xi) &= e^{-i\int_0^t\{\frac{1}{2}|\xi(s;t,x,\xi)|^2+\widetilde{V}(s,x(s;t,x,\xi))\}ds}W_{\varphi_0}u_0(x(0;t,x,\xi),\xi(0;t,x,\xi)) \\ &- i\int_0^t e^{-i\int_s^t\{\frac{1}{2}|\xi(s_1;t,x,\xi)|^2+\widetilde{V}(s_1,x(s_1;t,x,\xi))\}ds_1}Ru(s,x(s;t,x,\xi),\xi(s;t,x,\xi))ds, \end{aligned}$$

where $x(s;t,x,\xi)$ and $\xi(s;t,x,\xi)$ are the solutions of

$$\begin{cases} \dot{x}(s) &= \xi(s),\ x(t) = x, \\ \dot{\xi}(s) &= -x(s) - \nabla v(s,x(s)),\ \xi(t) = \xi. \end{cases}$$

For fixed t_0, we have

$$\begin{aligned} &W_{\varphi_\lambda(t-t_0)}u(t,x(t;t_0,x,\lambda\xi),\xi(t;t_0,x,\lambda\xi)) \\ &= e^{-i\int_0^t\{\frac{1}{2}|\xi(s;t_0,x,\lambda\xi)|^2+\widetilde{V}(s,x(s;t_0,x,\lambda\xi))\}ds}W_{\varphi_\lambda(-t_0)}u_0(x(0;t_0,x,\lambda\xi),\xi(0;t_0,x,\lambda\xi)) \\ &+i\int_0^t e^{-i\int_s^t\{\frac{1}{2}|\xi(s_1,t_0,x,\lambda\xi)|^2+\widetilde{V}(s_1,x(s_1;t_0,x,\lambda\xi))\}ds_1}Ru(s,x(s;t_0,x,\lambda\xi),\xi(s;t_0,x,\lambda\xi))ds, \end{aligned} \tag{12}$$

substituting $(x(t;t_0,x,\lambda\xi),\xi(t;t_0,x,\lambda\xi))$ and $\varphi_\lambda(-t_0,x)$ for (x,ξ) and $\varphi_0(x)$ respectively. Here we use the fact that $x(s;t,x(t;t_0,x,\lambda\xi),\xi(t;t_0,x,\lambda\xi)) = x(s;t_0,x,\lambda\xi)$, $\xi(s;t,x(t;t_0,x,\lambda\xi),\xi(t;t_0,x,\lambda\xi)) = \xi(s;t_0,x,\lambda\xi)$ and $e^{\frac{i}{2}t(\Delta-|x|^2)}\varphi_\lambda(-t_0,x) = \varphi_\lambda(t-t_0,x)$.

We only show the sufficiency here because the necessity is proved in the same way. To do so, we show that there exist a neighborhood K of x_0 and a conic neighborhood Γ of ξ_0 such that the following assertion $P(\sigma,\varphi_0)$ holds for all $\sigma\geq 0$

and for all $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)\backslash\{0\}$.
$P(\sigma, \varphi_0)$: " For $a \geq 1$ there exists a positive constant C_{σ,a,φ_0} such that

$$|W_{\varphi_\lambda(t-t_0)}u(t, x(t; t_0, x, \lambda\xi), \xi(t; t_0, x, \lambda\xi))| \leq C_{\sigma,a,\varphi_0}\lambda^{-\sigma} \tag{13}$$

for all $x \in K$, all $\xi \in \Gamma$ with $1/a \leq |\xi| \leq a$, all $\lambda \geq 1$ and $0 \leq t \leq t_0$. "

In fact, taking $t = t_0$, we have $\varphi_\lambda(t_0 - t_0) = \varphi_{0,\lambda}$, $x(t_0; t_0, x, \lambda\xi) = x$ and $\xi(t_0; t_0, x, \lambda\xi) = \lambda\xi$. Hence from (13), we have immediately

$$|W_{\varphi_\lambda}u(t_0, x, \lambda\xi)| \leq C_{\sigma,a,\varphi_0}\lambda^{-\sigma}$$

for $\lambda \geq 1$, $x \in K$ and $\xi \in \Gamma$ with $1/a \leq |\xi| \leq a$. This and Proposition 2.2 show the sufficiency.

We fix $b = (2-\rho)/4$. We write $x(s) = x(s; t_0, x, \lambda\xi), \xi(s) = \xi(s; t_0, x, \lambda\xi)$, $t(s) = s - t_0$ and $\varphi_\lambda(x) = (\varphi_0)_\lambda(x)$ for simple description.

We show by induction with respect to σ that $P(\sigma, \varphi_0)$ holds for all $\sigma \geq 0$ and for all $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)\backslash\{0\}$.

First we show that $P(0, \varphi_0)$ holds for all $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)\backslash\{0\}$. Since $u_0(x) \in L^2(\mathbb{R}^n)$, $u(t,x) \in C(\mathbb{R}; L^2(\mathbb{R}^n))$. Schwarz's inequality and conservativity for L^2 norm of solutions of (1) show that

$$\begin{aligned}\left|W_{\varphi_\lambda(t)}[u(t)](x, \lambda\xi)\right| &\leq \int |\varphi_\lambda(t, y-x)||u(t,y)|dy \\ &\leq \|\varphi_\lambda(t,\cdot)\|_{L^2}\|u(t,\cdot)\|_{L^2} \\ &= \|\varphi_\lambda(\cdot)\|_{L^2}\|u_0(\cdot)\|_{L^2} = \|\varphi_0(\cdot)\|_{L^2}\|u_0(\cdot)\|_{L^2}.\end{aligned}$$

Hence $P(0, \varphi_0)$ holds.

Next we show that for fixed $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)\backslash\{0\}$, $P(\sigma + b, \varphi_0)$ holds under the assumption that $P(\sigma, \varphi_0)$ holds for all $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)\backslash\{0\}$. To do so, it suffices to show that for fixed φ_0, there exists a positive constant C_{a,φ_0} such that

$$|Ru(s, x(s; t_0, x, \lambda\xi), \xi(s; t_0, x, \lambda\xi))| \leq C_{a,\varphi_0}\lambda^{-(\sigma+b)} \tag{14}$$

for all $x \in K$, all $\xi \in \Gamma$ with $1/a \leq |\xi| \leq a$, all $\lambda \geq 1$ and $0 \leq s \leq t_0$, since the first term of the right hand side of (12) is estimated by the condition on u_0.

Taylor's expansion of $v(s, x(s), y)$ yields that

$$\begin{aligned}&Ru(s, x(s), \xi(s), \lambda) \\ &= \sum_{2\leq|\alpha|\leq L-1} \frac{\partial_x^\alpha v(s, x(s))}{\alpha!} \int (x(s)-y)^\alpha \overline{\varphi_\lambda(s-t_0, y-x(s))} u(s,y) e^{-iy\xi(s)} dy + R_L,\end{aligned} \tag{15}$$

where

$$R_L(s,x(s),\xi(s),\lambda) = L\sum_{|\alpha|=L}\frac{1}{\alpha!}\frac{1}{\|\varphi_0\|_{L^2}^2}$$
$$\times\iint\Big(\int\int_0^1 \partial_x^\alpha v(s,x(s)-\theta(x(s)-y))(1-\theta)^{L-1}d\theta\,(x(s)-y)^\alpha$$
$$\times\overline{\varphi_\lambda(s-t_0,y-x(s))}\,\varphi_\lambda(s-t_0,y-z)e^{-iy(\xi(s)-\eta)}dy\Big)\,W_{\varphi_\lambda(s-t_0)}u(s,z,\eta)dzd\eta.$$

Here we use the inversion formula of the wave packet transform

$$\frac{1}{\|\varphi\|_{L^2}^2}W_\varphi^{-1}W_\varphi f(x) = f(x),$$

where

$$W_\varphi^{-1}f(x) = (2\pi)^{-n}\iint f(y,\xi)\varphi(y-x)e^{ix\xi}d\xi dy$$

for a smooth tempered function $f(y,\xi)$ on $\mathbb{R}^{2n}$.

The strategy for the proof of (14) is the following. In Step 1, we estimate the first term of the right hand side of (15). In Step 2, taking L sufficiently large according to the value of σ, we estimate the second term R_L of the right hand side of (15).

Without loss of generality, it suffices to show (15) for $0 < t_0 \le \pi$. Here we only show (14) in the case that $t_0 = \pi$. In the case that $0 < t_0 < \pi$, we can show (15) easier.

(Step1) We estimate the first term of the right hand side of (15). Since

$$x\varphi_\lambda(t,x) = e^{\frac{it}{2}(\Delta-|x|^2)}\left[(x\cos t - \sin t\nabla)\varphi_{0,\lambda}\right],$$

we have

$$\begin{aligned}
&(y-x(s))^\alpha\varphi_\lambda(t(s),y-x(s))\\
=&e^{\frac{it}{2}(\Delta-|x|^2)}\left[(x\cos t-\sin t\nabla)^\alpha\varphi_{0,\lambda}\right](t(s),x(s)-y)\\
=&\sum_{\beta+\gamma=\alpha}C_{\beta,\gamma}t(s)^{|\beta|}\lambda^{b(|\beta|-|\gamma|)}\varphi_\lambda^{(\beta,\gamma)}(t(s),y-x(s)),
\end{aligned}$$

where $\varphi^{(\beta,\gamma)}(x) = x^\gamma\partial_x^\beta\varphi_0(x)$ and $\varphi_\lambda^{(\beta,\gamma)}(t,x) = e^{\frac{i}{2}t(\Delta-|x|^2)}\left(\varphi^{(\beta,\gamma)}\right)_\lambda(x)$. Since

$$\begin{pmatrix}x(s)\\ \xi(s)\end{pmatrix} = \begin{pmatrix}\cos t(s) & \sin t(s)\\ -\sin t(s) & \cos t(s)\end{pmatrix}\begin{pmatrix}x\\ \lambda\xi\end{pmatrix}$$
$$+\int_{t_0}^s\begin{pmatrix}\cos(s-\tau) & \sin(s-\tau)\\ -\sin(s-\tau) & \cos(s-\tau)\end{pmatrix}\begin{pmatrix}0\\ -\nabla v(\tau,x(\tau))\end{pmatrix}d\tau,$$

we have for some $\lambda_0 \ge 1$ that

$$\frac{1}{2}\lambda|\xi||\sin t(s)| \le |x(s)| \le 2\lambda|\xi||\sin t(s)| \tag{16}$$

for $\lambda \geq \lambda_0$ and $\lambda^{-2b} \leq t(s) \leq \pi - \lambda^{-2b}$. Hence the assumption of induction yields that for $\lambda^{-2b} \leq s \leq t_0$

$$\begin{aligned}
&|(\text{The first term of the right hand side of (15)})| \\
&\leq \sum_{2\leq|\alpha|\leq L-1} \sum_{\beta+\gamma=\alpha} \frac{1}{\alpha!}|\partial_x^\alpha v(s,x(s))| C_{\beta,\gamma}|\sin t(s)|^{|\beta|}\lambda^{b(|\beta|-|\gamma|)} \left|W_{\varphi_\lambda^{(\beta,\gamma)}(t(s),x)}u(s,x(s),\xi(s))\right| \\
&\leq \sum_{2\leq|\alpha|\leq L-1} \sum_{\beta+\gamma=\alpha} \frac{1}{\alpha!} C(1+|x(s)|)^{\rho-|\alpha|} C_{\beta,\gamma}|\sin t(s)|^{|\beta|}\lambda^{b(|\beta|-|\gamma|)} C\lambda^{-\sigma} \\
&\leq \sum_{2\leq|\alpha|\leq L-1} \sum_{\beta+\gamma=\alpha} \frac{1}{\alpha!} C(1+|\sin t(s)|\lambda)^{\rho-|\alpha|} C_{\beta,\gamma}|\sin t(s)|^{|\beta|}\lambda^{b(|\beta|-|\gamma|)} C\lambda^{-\sigma} \\
&\leq C' \sum_{2\leq|\alpha|\leq L-1} \lambda^{\rho-|\alpha|+b|\alpha|-\sigma} \\
&\leq C''\lambda^{\rho-2+2b-\sigma} = C''\lambda^{-(2-\rho)/2-\sigma},
\end{aligned}$$

since $b = (2-\rho)/4$.

For $0 \leq t(s) \leq \lambda^{-2b}$, we have

$$\begin{aligned}
&|(\text{The first term of the right hand side of (15)})| \\
&\leq \sum_{2\leq|\alpha|\leq L-1} \sum_{\beta+\gamma=\alpha} \frac{1}{\alpha!}|\partial_x^\alpha v(s,x(s))| C_{\beta,\gamma}|\sin t(s)|^{|\beta|}\lambda^{b(|\beta|-|\gamma|)} \left|W_{\varphi_\lambda^{(\beta,\gamma)}(t(s),x)}u(s,x(s),\xi(s))\right| \\
&\leq \sum_{2\leq|\alpha|\leq L-1} \sum_{\beta+\gamma=\alpha} \frac{1}{\alpha!} C(1+|x(s)|)^{\rho-|\alpha|} C_{\beta,\gamma}|\sin t(s)|^{|\beta|}\lambda^{b(|\beta|-|\gamma|)} C\lambda^{-\sigma} \\
&\leq \sum_{2\leq|\alpha|\leq L-1} \sum_{\beta+\gamma=\alpha} \frac{1}{\alpha!} C(1+|\sin t(s)|\lambda)^{\rho-|\alpha|} C_{\beta,\gamma}|\sin t(s)|^{|\beta|}\lambda^{b(|\beta|-|\gamma|)} C\lambda^{-\sigma} \\
&\leq C' \sum_{2\leq|\alpha|\leq L-1} \lambda^{-2b|\beta|+b|\beta|-\sigma} \\
&\leq C''\lambda^{-2b-\sigma}.
\end{aligned}$$

(Step 2) We estimate R_L. Let ψ_1, ψ_2 be C^∞ functions on $\mathbb{R}$ satisfying

$$\psi_1(s) = \begin{cases} 1 & \text{for } s \leq 1, \\ 0 & \text{for } s \geq 2, \end{cases}$$

$$\psi_2(s) = \begin{cases} 0 & \text{for } s \leq 1, \\ 1 & \text{for } s \geq 2, \end{cases}$$

$$\psi_1(s) + \psi_2(s) = 1 \quad \text{for all } s \in \mathbb{R}.$$

Take $d > 0$ satisfying $1 - b < d < 1$. Putting $v_\alpha(s,x(s),y) = \int_0^1 \partial_x^\alpha v(s,x(s) -$

$\theta(x(s)-y))(1-\theta)^{L-1}d\theta$ and

$$
\begin{aligned}
&I_{\alpha,j}(s,x(s),\xi(s),\lambda)\\
&= \iiint \psi_j\left(\frac{|y-x(s)|}{(1+\lambda|\sin t(s)|)^{2b}\lambda^{d-1}}\right) v_\alpha(s,x(s),y)(x(s)-y)^\alpha\\
&\qquad \overline{\varphi_\lambda(t(s),y-x(s))}\varphi_\lambda(t(s),y-z)W_{\varphi_\lambda(t(s))}u(s,z,\eta)e^{-iy(\xi(s)-\eta)}dzd\eta dy
\end{aligned}
$$

for $j=1,2$, we have

$$
R_L(s,x(s),\xi^*,\lambda) = L\sum_{|\alpha|=L}\frac{1}{\alpha!}\frac{1}{\|\varphi_0\|_{L^2}^2}\sum_{j=1}^{2} I_{\alpha,j}(s,x(s),\xi(s),\lambda). \tag{17}
$$

We need to show that for $j=1,2$, there exists a positive constant C_{σ,a,φ_0} such that

$$
|I_{\alpha,j}(s,x(s),\xi(s),\lambda)| \le C_{\sigma,a,\varphi_0}\lambda^{-\sigma-(2-\rho)/2} \tag{18}
$$

for $\lambda \ge 1$, $x\in K$, $\xi \in \Gamma$ with $1/a \le |\xi| \le a$ and $0\le s\le t_0$.

First we estimate $I_{\alpha,1}$. For $\lambda^{-2b} \le t(s) \le \pi - \lambda^{-2b}$, integration by parts and the fact that $(1-\triangle_y)e^{iy(\xi-\eta)} = (1+|\xi-\eta|^2)e^{iy(\xi-\eta)}$ yield that

$$
\begin{aligned}
&I_{\alpha,1}(s,x(s),\xi(s),\lambda) = \iiint \left(1+|\xi-\eta|^2\right)^{-N}\\
&\times(1-\triangle_y)^N\left[\overline{\varphi_\lambda(s-t_0,y-x(s))}\varphi_\lambda(s-t_0,y-z)\psi_1\left(\frac{|y-x(s)|}{(1+\lambda|\sin t(s)|)^{2b}\lambda^{d-1}}\right)\right.\\
&\qquad\left.\times v_\alpha(s,x(s),y)\right]e^{iy(\xi-\eta)}W_{\varphi_\lambda(s-t_0)}u(s,z,\eta)e^{-iy(\xi(s)-\eta)}dyd\eta dz.
\end{aligned}
$$

Since $|y-x(s)| \le C(1+\lambda|\sin t(s)|)^{2b}\lambda^{d-1}$ in the support of $\psi_1\left(\frac{|y-x(s)|}{(1+\lambda|\sin t(s)|)^{2b}\lambda^{d-1}}\right)$ with respect to y, the estimate (16) shows that

$$
\begin{aligned}
&|\partial_x^\alpha v(s,x(s)+\theta(y-x(s)))||(x(s)-y)^\alpha|\\
&\le C(1+|x(s)+\theta(y-x(s))|)^{\rho-L}(1+\lambda|t(s)|)^{2bL}\lambda^{(d-1)L}\\
&\le C(1+|x(s)|-|y-x(s)|)^{\rho-L}(1+\lambda|\sin t(s)|)^{2bL}\lambda^{(d-1)L}\\
&\le C(1+\lambda|\sin t(s)|)^{\rho-(1-2b)L}\lambda^{(d-1)L},
\end{aligned}
$$

from which we have

$$
|I_{\alpha,1}(s,x(s),\xi(s),\lambda)| \le C\lambda^{(d-1)L}\lambda^{l},
$$

where l are positive numbers which are independent of L. Since $d-1<0$, (18) with $j=1$ holds if we take L sufficiently large.

For $0\le t(s)\le \lambda^{-2b}$ or $\pi-\lambda^{-2b}\le t(s)\le\pi$, we have $|\sin t(s)| \le \lambda^{-2b}$. Hence $|y-x(s)| \le C(1+\lambda^{1-2b})^{2b}\lambda^{d-1} \le C'\lambda^{d-1}$, which shows (15) if we take L sufficiently large.

Finally we estimate $I_{\alpha,2}$. Before the estimate, we estimate $\varphi_\lambda(t,x) = e^{\frac{i}{2}t(\Delta-|x|^2)}\varphi_\lambda(x) = U(t)\varphi_{0,\lambda}$ in the domain $D = \{(t,x)||x| \geq (1+|\lambda \sin t|)^{2b}\lambda^{d-1}\}$. Since $\partial_{x_j}U(t) = U(t)(\cos t\partial_{x_j} - i(\sin t)x_j)$ and $x_jU(t) = U(t)(-i\sin t\partial_{x_j} + (\cos t)x_j)$,

$$
\begin{aligned}
&|(1+|x|^2)^M \partial_{x_j}^\alpha \varphi_\lambda(t,x)| \\
=&|U(t)(1+|x\cos t - i\sin t\nabla|^2)^M(\cos\partial_x - i(\sin t)x)^\alpha\varphi_\lambda| \\
\leq& \sum_{|\gamma_1+\gamma_2|\leq 2M}\sum_{\alpha_1+\alpha_2\leq\alpha} \lambda^{b(-|\gamma_1|-|\alpha_1|+|\alpha_2|)}(\lambda^b|\sin t|)^{|\gamma_2|}|U(t)(x^{\gamma_1+\alpha_2}\partial_x^{\gamma_2+\alpha_1}\varphi)_\lambda|,
\end{aligned}
$$

which yields that

$$
\begin{aligned}
|I_{\alpha,2}| \leq& \iiint (1+|\eta-\xi(s)|^2)^{-N} \\
&\times |(1-\Delta_y)^N\left[\varphi_2 v_\alpha(x(s)-y)^\alpha\overline{\varphi_\lambda(t_0-s,y-x(s))}\varphi_\lambda(t_0-s,y-z)\right]| \\
&\qquad \times |W_{\varphi_\lambda(t_0-s)}u(s,z,\eta)|dzd\eta dy \\
\leq& \sum_{|\alpha_1+\ldots+\alpha_4|\leq N}\sum_{\substack{\beta_1+\beta_2\leq\alpha_3\\ \beta_2+\beta_3\leq\alpha_4}} C\iiint (1+|\eta-\xi(s)|^s)^{-N}|\partial_y^{\alpha_1}\varphi_2||\partial_y^{\alpha_2}(v_\alpha(x(s)-y)^\alpha)| \\
&\times C(1+|y-x(s)|^2)^{-M}\sum_{|\gamma_1+\gamma_2|\leq 2M}\lambda^{b(-|\gamma_1|-|\beta_1|+|\beta_2|)}\left(\lambda^b|\sin t(s)|\right)^{|\gamma_2|} \\
&\qquad \times \left|U(t_0-s)\left(x^{\gamma_1+\beta_2}\partial_x^{\gamma_2+\beta_1}\varphi_0\right)_\lambda\right|(y-x(s)) \\
&\times C\lambda^{b(-|\beta_3|+|\beta_4|)}\left|U(t_0-s)\left(x^{\beta_3}\partial_x^{\beta_4}\varphi_0\right)_\lambda\right|(y-z)|W_{\varphi_\lambda(t_0-s)}u(s,z,\eta)|dzd\eta dy.
\end{aligned}
\tag{19}
$$

Since $b+d-1>0$ and $1-2b>0$, we have

$$
\begin{aligned}
&(1+|y-x(s)|^2)^{-M}\left(\lambda^b|\sin t(s)|\right)^{|\gamma_2|}|y-x(s)|^{|\alpha|} \\
&\leq (1+|y-x(s)|^2)^{-n}\left(1+(1+\lambda|\sin t(s)|)^{4b}\lambda^{2(d-1)}\right)^{-M+L+n}\left(\lambda^b|\sin t(s)|\right)^{2M} \\
&\leq (1+|y-x(s)|^2)^{-n}\lambda^{-2M(b+d-1)+2(L+n)(2b+d-1)}|\sin t(s)|^{2M(1-2b)} \\
&\leq (1+|y-x(s)|^2)^{-n}\lambda^{-2M(b+d-1)+2(L+n)(2b+d-1)}.
\end{aligned}
$$

This and (19) shows by Schwarz's inequality that

$$
\begin{aligned}
|I_{\alpha,2}| \leq& C\sum_{|\alpha_1+\ldots+\alpha_4|\leq N}\sum_{\substack{\beta_1+\beta_2\leq\alpha_3\\ \beta_2+\beta_3\leq\alpha_4}}\sum_{|\gamma_1+\gamma_2|\leq 2M}\lambda^{-2M(b+d-1)+2(L+n)(2b+d-1)}\|(1+|\cdot|^2)^{-n}\|_{L^2_y} \\
&\times \lambda^{b(|\alpha_3|+|\alpha_4|)}\|(1+|\cdot|^2)^{-n}\|_{L^2_\eta}\left\|U(t_0-s)\left(x^{\gamma_1+\beta_2}\partial_x^{\gamma_2+\beta_1}\varphi_0\right)_\lambda\right\|_{L^2_y} \\
&\times \left\|U(t_0-s)\left(x^{\beta_3}\partial_x^{\beta_4}\varphi_0\right)_\lambda\right\|_{L^2_z}\|W_{\varphi_\lambda(t_0-s)}u(s,z,\eta)\|_{L^2_{z,\eta}(\mathbb{R}^{2n})} \\
\leq& C\lambda^{-2M(b+d-1)+2(L+n)(2b+d-1)+2bN}.
\end{aligned}
$$

Hence $|I_{2,\alpha}| \le C\lambda^{-\sigma-b}$ is valid in $\lambda^{-2b} \le t(s) \le t_0$, since M can be chosen arbitrary.
For $0 \le t(s) \le \lambda^{-2b}$ or $\pi - \lambda^{-2b} \le t(s) \le \pi$, we have

$$\begin{aligned}
&(1+|y-x(s)|^2)^{-M}\left(\lambda^b|\sin t(s)|\right)^{|\gamma_2|}|y-x(s)|^{|\alpha|}\\
&\le (1+|y-x(s)|^2)^{-n}\left(1+(1+\lambda|\sin t(s)|)^{4b}\lambda^{2(d-1)}\right)^{-M+L+n}\left(\lambda^b|\sin t(s)|\right)^{2M}\\
&\le (1+|y-x(s)|^2)^{-n}\left(1+(1+\lambda^{1-2b})^{4b}\lambda^{2(d-1)}\right)^{-M+L+n}\left(\lambda^{-b}\right)^{2M}\\
&\le C(1+|y-x(s)|^2)^{-n}\lambda^{2(d-l)(-M+L+n)},
\end{aligned}$$

which shows that $|I_{2,\alpha}| \le C\lambda^{-\sigma-b}$ is valid if we take M sufficiently large. □

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